

SINGULAR LOCUS OF INSTANTON SHEAVES ON  $\mathbb{P}^3$ 

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ABSTRACT. We prove that the singular locus of rank 2 instanton sheaf  $E$  on  $\mathbb{P}^3$  which is not locally free has pure dimension 1. Moreover, we also show that the dual and double dual of  $E$  are isomorphic locally free instanton sheaves, and that the sheaves  $\mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^3})$  and  $E^{\vee\vee}/E$  are rank 0 instantons. We also provide explicit examples of instanton sheaves of rank 3 and 4 illustrating that all of these claims are false for higher rank instanton sheaves.

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## 1. INTRODUCTION

Mathematical instanton bundles have been intensely studied by several authors since its introduction in the late 1970's. They derive their nomenclature from Gauge Theory: mathematical instanton bundles on odd dimensional complex projective spaces  $\mathbb{CP}^{2k+1}$  are precisely those holomorphic vector bundles that arise, via the twistor correspondence, in relation with quaternionic instantons on quaternionic projective spaces  $\mathbb{HP}^k$ , see [9] for details, and also [11].

The simplest case of such objects are rank 2 instanton bundles on  $\mathbb{CP}^3$ , and there is a vast literature about them. One outstanding problem that resisted solution for a couple of decades regards the irreducibility and smoothness of the moduli space  $\mathcal{I}(c)$  of rank 2 instanton bundles on  $\mathbb{CP}^3$  of charge  $c$ . It was known since 2003 that  $\mathcal{I}(c)$  is smooth and irreducible for  $c \leq 5$ , see [1] and the references therein. Recently, Tikhomirov has proved in [14, 15] irreducibility for arbitrary  $c$ ; while the second named author and Verbitsky established smoothness for every  $c$ , see [8].

The next step is to understand how instanton bundles degenerate, that is to study non locally free instanton sheaves. Maruyama and Trautmann were the first to consider *limits of instantons* on  $\mathbb{P}^3$  in [10]; in [6] instanton sheaves on arbitrary projective spaces  $\mathbb{P}^n$  are studied. In this paper we only consider instanton sheaves on  $\mathbb{P}^3$ ; these are defined as torsion free sheaves  $E$  on  $\mathbb{P}^3$  with  $c_1(E) = 0$ ,  $c_3(E) = 0$ , and satisfying

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0,$$

compare with [10, 1.1, page 216] and [6, page 69]. The charge of  $E$  is given by its second Chern class  $c_2(E)$ .

The goal of this paper is to study the singular locus of rank 2 instanton sheaves, showing that they are of pure dimension 1. In the process, the rank 0 instantons introduced by Hauzer and Langer in [4, Definition 6.1] also appear. A rank 0 instanton is a pure dimension 1 sheaf  $Z$  on  $\mathbb{P}^3$  such that  $h^0(Z(-2)) = h^1(Z(-2)) = 0$ ;  $d := h^0(Z(-1))$  is called the degree of  $Z$ ; see Section 3.2 below for details. We say that two rank 0 instantons  $Z_1$  and  $Z_2$  are dual to each other if  $\mathcal{E}xt^2(Z_1, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_2$  and  $\mathcal{E}xt^2(Z_2, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_1$ .

More precisely, with the previous definitions in mind, we prove the following.

**Main Theorem.** *If  $E$  is a rank 2 non locally free instanton sheaf on  $\mathbb{P}^3$ , then*

- (i) *its singular locus has pure dimension 1;*
- (ii)  *$E^\vee$  and  $E^{\vee\vee}$  are isomorphic locally free instanton sheaves;*
- (iii) *the sheaves  $\mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^3})$  and  $E^{\vee\vee}/E$  are dual rank 0 instantons of degree  $c_2(E) - c_2(E^{\vee\vee})$ .*

*In addition, if  $c_2(E) - c_2(E^{\vee\vee}) = 1$ , then the singular locus of  $E$  consists of a single line.*

All of these claims are false for instanton sheaves of higher rank. Indeed, we provide explicit examples of instanton sheaves of rank 3 and 4 on  $\mathbb{P}^3$  for which  $E^\vee$  and  $E^{\vee\vee}$  are not locally free and not instanton sheaves, respectively, see Section 3.3. In addition, we also describe rank 3 instanton sheaves whose singular loci are either of dimension 0 or not of pure dimension 1, see Section 5.2 below.

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## 2. SHEAVES AND MONADS

In this section, we fix the notation that will be used in this work and we recall some basic definitions and results. We work over a fixed algebraically closed field  $\kappa$  of characteristic zero. By a projective variety  $X$  we understand a nonsingular, projective, integral, separated noetherian scheme of finite type over  $\kappa$ . All sheaves on  $X$  are coherent sheaves of  $\mathcal{O}_X$ -modules.

**2.1. Sheaves.** Let  $E$  be a coherent sheaf on projective variety  $X$ ; its *support* is the closed set

$$(1) \quad \text{Supp}(E) = \{x \in X \mid E_x \neq 0\},$$

where  $E_x$  denotes the stalk of  $E$  over the point  $x \in X$ . The dimension of  $E$ , denoted by  $\dim(E)$ , is defined to be the dimension  $\text{Supp}(E)$  as an algebraic set. A coherent

sheaf  $E$  is said to be of pure dimension  $d$  if  $\dim E = d$  and every nonzero subsheaf of  $E$  also has dimension  $d$ .

The *singular locus* of  $E$  is defined as the following closed set

$$(2) \quad \begin{aligned} \text{Sing}(E) &= \{x \in X \mid E_x \text{ not free } \mathcal{O}_{X,x} \text{ - module}\} \\ &= \bigcup_{p=1}^{\dim X} \text{Supp } \mathcal{E}xt^p(E, \mathcal{O}_X). \end{aligned}$$

Recall also that  $E$  is *torsion free* if the canonical map into the double dual sheaf  $E^{\vee\vee}$  is injective; if such map is an isomorphism, then  $E$  is *reflexive*. One can show that if  $E$  is torsion free, then  $\text{codim Sing}(E) \geq 2$ , and if  $E$  is reflexive, then  $\text{codim Sing}(E) \geq 3$ .

Two sheaves derived from  $E$  will play important parts later on,

$$(3) \quad S_E := \mathcal{E}xt^1(E, \mathcal{O}_X) \text{ and } Q_E := E^{\vee\vee}/E.$$

Clearly,  $\text{Supp}(S_E) \subseteq \text{Sing}(E)$ ; note also that  $\text{Supp}(Q_E) \subseteq \text{Sing}(E)$ . Indeed, if  $x \notin \text{Sing}(E)$ , then  $E_x$  is free as an  $\mathcal{O}_{X,x}$ -module, hence  $E_x \simeq (E^{\vee\vee})_x$  and  $x \notin \text{Supp}(Q_E)$ . Moreover, it is not difficult to see that if  $E^{\vee\vee}$  is locally free, then in fact  $\text{Supp}(Q_E) = \text{Sing}(E)$ .

Hartshorne proves in [3] the following results that will play key roles in the proof of our main results.

**Proposition 1.** ([3, Prop. 1.10]) *If  $F$  is a rank 2 reflexive sheaf on  $\mathbb{P}^3$ , then  $F^\vee \cong F \otimes (\det F)^{-1}$ .*

**Proposition 2.** ([3, Prop. 2.5]) *If  $F$  is a reflexive sheaf on  $\mathbb{P}^3$ , then there are isomorphisms*

$$H^0(F^\vee(-4)) \simeq H^3(F)^\vee, \quad H^3(F^\vee(-4)) \simeq H^0(F)^\vee$$

*and an exact sequence*

$$0 \rightarrow H^1(F^\vee(-4)) \rightarrow H^2(F)^\vee \rightarrow H^0(S_F(-4)) \rightarrow H^2(F^\vee(-4)) \rightarrow H^1(F)^\vee \rightarrow 0.$$

Another key ingredient is the following criterion due to Roggero.

**Proposition 3.** (cf. [13, Thm. 2.3]) *If  $F$  is a rank 2 reflexive sheaf with  $c_1(F) = 0$  on  $\mathbb{P}^3$ , then  $F$  is locally free if and only if  $h^2(F(p)) = 0$  for some  $p \leq -2$ .*

**2.2. Monads.** Recall that a *monad* on  $X$  is a complex  $M^\bullet$  of locally free sheaves on  $X$  of the following form:

$$(4) \quad M_\bullet : A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

which is exact on the first and last terms. The sheaf  $E = \ker \beta / \text{Im } \alpha$  is called the cohomology of the monad  $M^\bullet$ .

The *degeneration locus* of the monad (4) consists of the following set

$$(5) \quad \Delta(M_\bullet) = \{x \in X \mid \alpha(x) \text{ is not injective}\}.$$

**Lemma 4.** *If  $E$  is the cohomology sheaf of a monad  $M^\bullet$  as in (4), then  $\Delta(M_\bullet) = \text{Supp}(S_E) = \text{Sing}(E)$ . In particular,  $\text{Supp}(Q_E) \subseteq \text{Supp}(S_E)$ .*

*Proof.* One can break the monad (4) into the short exact sequences

$$(6) \quad 0 \rightarrow K \rightarrow B \xrightarrow{\beta} C \rightarrow 0$$

where  $K := \ker \beta$ , and

$$(7) \quad 0 \rightarrow A \xrightarrow{\alpha} K \rightarrow E \rightarrow 0$$

Dualizing (7) we obtain

$$(8) \quad 0 \rightarrow E^\vee \rightarrow K^\vee \xrightarrow{\alpha^\vee} A^\vee \rightarrow \mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^3}) \rightarrow 0$$

and  $\mathcal{E}xt^p(E, \mathcal{O}_{\mathbb{P}^3}) = 0$  for  $p \geq 2$ . It then follows immediately that  $\text{Supp}(S_E) = \text{Sing}(E)$ . To see that  $\Delta(M_\bullet) = \text{Supp}(S_E)$ , note that  $x \in \text{Supp}(S_E)$  if and only if the map of stalks  $(\alpha^\vee)_x$  is not surjective; this happens if and only if the map of fibers  $\alpha^\vee(x)$  is not surjective, which is equivalent to  $x \in \Delta(M_\bullet)$ .  $\square$

In general, the dual of a monad may not be a monad, only a complex of the form

$$(9) \quad M_\bullet^\vee : C^\vee \xrightarrow{\beta^\vee} B^\vee \xrightarrow{\alpha^\vee} A^\vee$$

whose first map is injective. It may have two nontrivial cohomology sheaves:  $\mathcal{H}^0(M_\bullet^\vee) := \ker(\alpha^\vee)/\text{Im}(\beta^\vee)$  and  $\mathcal{H}^1(M_\bullet^\vee) := \text{coker}(\alpha^\vee)$ .

**Lemma 5.** *If  $E$  is the cohomology sheaf of a monad  $M^\bullet$  of the form (4), then  $E^\vee \simeq \mathcal{H}^0(M_\bullet^\vee)$  and  $S_E := \mathcal{H}^1(M_\bullet^\vee)$ .*

*Proof.* Dualizing (6) and breaking (8) into short exact sequences, we obtain the following three short exact sequences:

$$(10) \quad 0 \rightarrow C^\vee \xrightarrow{\beta^\vee} B^\vee \rightarrow K^\vee \rightarrow 0,$$

$$(11) \quad 0 \rightarrow E^\vee \rightarrow K^\vee \xrightarrow{\alpha^\vee} T \rightarrow 0,$$

where  $T = \text{Im}(\alpha^\vee)$ , and

$$(12) \quad 0 \rightarrow T \rightarrow A^\vee \rightarrow S_E \rightarrow 0.$$

On the other hand, a complex of the form (9) whose first map is injective can be broken down into three short exact sequences as follows

$$(13) \quad 0 \rightarrow C^\vee \xrightarrow{\beta^\vee} B^\vee \rightarrow V \rightarrow 0,$$

where  $V := \text{coker}(\beta^\vee)$ ,

$$(14) \quad 0 \rightarrow \mathcal{H}^0(M_\bullet^\vee) \rightarrow V \xrightarrow{\alpha^\vee} T \rightarrow 0,$$

where  $T = \text{Im}(\alpha^\vee)$ , and

$$(15) \quad 0 \rightarrow T \rightarrow A^\vee \rightarrow \mathcal{H}^1(M_\bullet^\vee) \rightarrow 0.$$

The desired conclusion follows from the comparison of the two sets of sequences.  $\square$

We have one more observation regarding reflexive sheaves on 3-dimensional varieties.

**Lemma 6.** *Let  $M_\bullet$  be a monad as in equation (4) whose degeneration locus has codimension at least 3. Then its cohomology sheaf  $E$  is reflexive.*

*Proof.* Dualizing sequence (12) we conclude that  $T^\vee \simeq A$  and  $\mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3}) = \mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3}) = 0$ , because  $S_E$  is supported in codimension at least 3. Then dualizing sequence (11) we obtain, since  $K$  is locally free:

$$(16) \quad 0 \rightarrow A \xrightarrow{\alpha^{\vee\vee}} K \simeq E^{\vee\vee} \rightarrow 0;$$

in other words,  $E^{\vee\vee} \simeq \text{coker } \alpha^{\vee\vee} \simeq \text{coker } \alpha \simeq E$ , as desired.  $\square$

### 3. INSTANTONS ON $\mathbb{P}^3$

We are finally in position to focus on our main object of study. In this Section, we review the definitions of instanton sheaves and rank 0 instantons on  $\mathbb{P}^3$  and prove new basic results regarding their structure.

**3.1. Instanton sheaves.** Recall from [6, p. 69] that an *instanton sheaf* on  $\mathbb{P}^3$  is a torsion free coherent sheaf  $E$  with  $c_1(E) = 0$  and

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

Moreover, the integer  $c := h^1(E(-1))$  is called the *charge* of  $E$ . One can check that it coincides with  $c_2(E)$ .

As observed in the Introduction, locally free instanton sheaves of rank 2 are precisely (*mathematical*) *instanton bundles*. In fact, one can show that instanton sheaves are precisely those that can be obtained as the cohomology of a *linear monad* of the form

$$(17) \quad \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c},$$

where  $r$  is the rank of  $E$ .

If  $E$  is a locally free instanton sheaf on  $\mathbb{P}^3$ , then it is easy to see that its dual  $E^\vee$  and its double dual  $E^{\vee\vee}$  are also instanton sheaves. The same is not true in general; in fact, we will see below that if  $E$  is a reflexive instanton sheaf which is not locally free, then  $E^\vee$  is not instanton. However, the sheaves  $E^\vee$  and  $E^{\vee\vee}$  still retains the following properties.

**Lemma 7.** *If  $E$  is an instanton sheaf on  $\mathbb{P}^3$ , then*

- (i)  $h^0(E^\vee(-1)) = h^1(E^\vee(-2)) = 0$ ;
- (ii)  $h^2(E^\vee(-2)) = h^0(S_E(-2))$ ;
- (iii)  $h^2(E^{\vee\vee}(-2)) = h^3(E^{\vee\vee}(-3)) = 0$ ;
- (iv)  $h^1(E^{\vee\vee}(-2)) = h^1(Q_E(-2))$ .

*If, in addition,  $E$  is either  $\mu$ -semistable or of trivial splitting type, then  $h^3(E^\vee(-3)) = h^0(E^{\vee\vee}(-1)) = 0$ .*

*Proof.* Dualizing the monad (17) we obtain the complex

$$(18) \quad \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \xrightarrow{\beta^\vee} \mathcal{O}_{\mathbb{P}^3}^{\oplus r+2c} \xrightarrow{\alpha^\vee} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c}.$$

We know from Lemma 5 that  $E^\vee$  is the 0<sup>th</sup> cohomology of (18), while its first cohomology yields the sheaf  $S_E$ . Breaking (18) into short exact sequences as in the proof of Lemma 5 and passing to cohomology, we obtain that (i) and (ii).

Since  $E$  is torsion free, we know that  $\dim \text{Sing}(E) \leq 1$ ; since  $\text{Supp}(Q_E) \subseteq \text{Sing}(E)$ , we conclude that  $\dim Q_E \leq 1$ . Now we use the short exact sequence

$$(19) \quad 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q_E \rightarrow 0$$

to obtain (iii) and (iv), since  $h^2(Q_E(k)) = h^3(Q_E(k)) = 0$  for every  $k \in \mathbb{Z}$ .

Finally, if  $E$  is either  $\mu$ -semistable or of trivial splitting type, then the double dual sheaf  $E^{\vee\vee}$  is, respectively, either  $\mu$ -semistable or of trivial splitting type; in either case, we have that  $h^0(E^{\vee\vee}(-1)) = 0$ . The vanishing of  $h^3(E^\vee(-3))$  then follows from Lemma 2.  $\square$

The last result in this section describes the sheaves  $\mathcal{E}xt^p(S_E, \mathcal{O}_{\mathbb{P}^3})$  when  $E$  is an instanton sheaf. It also provides, in particular, a relation between the sheaves  $S_E$  and  $Q_E$ .

**Lemma 8.** *If  $E$  is an instanton sheaf on  $\mathbb{P}^3$ , then*

- (i)  $\mathcal{E}xt^1(S_E, \mathcal{O}_{\mathbb{P}^3}) = 0$ ;
- (ii)  $\mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq Q_E$ ;
- (iii)  $\mathcal{E}xt^3(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq S_{E^\vee}$ .

*Proof.* Dualizing the sequence

$$(20) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \xrightarrow{\alpha} K \rightarrow E \rightarrow 0$$

and breaking into short exact sequences we obtain

$$(21) \quad 0 \rightarrow E^\vee \rightarrow K^\vee \xrightarrow{\alpha^\vee} T \rightarrow 0,$$

where  $T = \text{Im}(\alpha^\vee)$ , and

$$(22) \quad 0 \rightarrow T \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c} \rightarrow S_E \rightarrow 0,$$

We can then gather (20) and the dual of (21) into the following diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} & \xrightarrow{\alpha} & K & \longrightarrow & E \longrightarrow 0 \\
 & & & & \downarrow \simeq & & \downarrow \\
 0 & \longrightarrow & T^\vee & \xrightarrow{\alpha^{\vee\vee}} & K & \longrightarrow & E^{\vee\vee} \longrightarrow S_T \longrightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & Q_E \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

where we recall that  $S_T = \mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3})$ .

It follows that  $T^\vee \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c}$  and  $\mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3}) \simeq Q_E$ . Note also that  $\mathcal{E}xt^2(T, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^1(E^\vee, \mathcal{O}_{\mathbb{P}^3})$  and  $\mathcal{E}xt^3(T, \mathcal{O}_{\mathbb{P}^3}) = 0$ .

Dualizing (22), it follows that  $\mathcal{E}xt^1(S_E, \mathcal{O}_{\mathbb{P}^3}) = 0$ ,  $\mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^1(T, \mathcal{O}_{\mathbb{P}^3})$  and  $\mathcal{E}xt^3(S_E, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(T, \mathcal{O}_{\mathbb{P}^3})$ .  $\square$

**3.2. Rank 0 instantons.** The notion of rank 0 instanton on  $\mathbb{P}^3$  was introduced in [4, Defn. 6.1]. A coherent sheaf  $Z$  on  $\mathbb{P}^3$  is called a *rank 0 instanton* if it has pure dimension 1 and  $h^0(Z(-2)) = h^1(Z(-2)) = 0$ . The integer  $d := h^0(Z(-1))$  is called the *degree* of  $Z$ .

One can show, see [4, Lemma 6.2], that  $Z$  is a rank 0 instanton if and only if there is a complex of the form (a.k.a. a *perverse instanton*)

$$Z_\bullet : \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d}$$

such that  $\mathcal{H}^{-1}(Z_\bullet) = \mathcal{H}^0(Z_\bullet) = 0$  and  $\mathcal{H}^1(Z_\bullet) = Z$ ; here,  $d$  is precisely the degree of  $Z$ .

In other words,  $Z$  is a rank 0 instanton of degree  $d$  if and only if it admits a resolution of the form

$$(23) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \rightarrow Z \rightarrow 0.$$

It follows immediately that  $\mathcal{E}xt^3(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$ . Note also that  $\mathcal{E}xt^1(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$ , since  $\text{codim } Z = 2$ .

**Lemma 9.** *If  $Z$  is a rank 0 instanton, then so is  $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$ .*

*Proof.* Break (23) into short exact sequences to obtain

$$(24) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\tau} I \rightarrow 0,$$

where  $I = \text{Im } \tau = \text{coker } \sigma$ , and

$$(25) \quad 0 \rightarrow I \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \rightarrow Z \rightarrow 0.$$

Dualizing (25), we conclude that  $I^\vee \simeq \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d}$ , since  $\mathcal{E}xt^1(Z, \mathcal{O}_{\mathbb{P}^3}) = 0$ , and  $\mathcal{E}xt^1(I, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$ .

Thus dualizing (24) we obtain

$$(26) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus d} \xrightarrow{\tau^\vee} \mathcal{O}_{\mathbb{P}^3}^{\oplus 2d} \xrightarrow{\sigma^\vee} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus d} \rightarrow \mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3}) \rightarrow 0,$$

thus  $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$  is a rank 0 instanton as well.  $\square$

As seen on the proof above,  $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$  is obtained essentially by dualizing the resolution that defines  $Z$ . For this reason, we say that  $\mathcal{E}xt^2(Z, \mathcal{O}_{\mathbb{P}^3})$  is the *dual* of  $Z$ . Two rank 0 instantons  $Z_1$  and  $Z_2$  are dual to each other if  $\mathcal{E}xt^2(Z_1, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_2$  and  $\mathcal{E}xt^2(Z_2, \mathcal{O}_{\mathbb{P}^3}) \simeq Z_1$ .

Finally, we now analyze whether the sheaves  $S_E$  and  $Q_E$  are rank 0 instantons. Note that  $\dim S_E \leq 1$  and  $\dim Q_E \leq 1$ , since  $E$  is torsion free; however, both sheaves may have zero dimensional subsheaves. Our next two results provide sufficient conditions for  $S_E$  and  $Q_E$  to be rank 0 instantons.

**Lemma 10.** *If  $E$  is an instanton sheaf which is not locally free and such that  $E^\vee$  is instanton, then  $S_E$  is rank 0 instanton.*

*Proof.* If  $E^\vee$  is instanton, then  $h^0(S_E(-2)) = h^0(E^\vee(-2)) = 0$ , by Lemma 7, item (ii); in particular,  $\text{Supp}(S_E)$  cannot have zero dimensional subsheaves, so it has pure dimension 1.

To see that  $h^1(S_E(-2)) = 0$ , note that the sequences (21) and (22) yields

$$h^1(S_E(-2)) = h^2(T(-2)) = h^3(E^\vee(-2)) = 0,$$

since  $E^\vee$  is instanton.  $\square$

**Lemma 11.** *If  $E$  is an instanton sheaf on  $\mathbb{P}^3$  which is not reflexive and such that  $E^{\vee\vee}$  is instanton, then  $Q_E$  is rank 0 instanton. Moreover, there is a short exact sequence*

$$(27) \quad 0 \rightarrow S_{E^{\vee\vee}} \rightarrow S_E \rightarrow \mathcal{E}xt^2(Q_E, \mathcal{O}_{\mathbb{P}^3}) \rightarrow 0.$$

*Proof.* Consider the sequence

$$(28) \quad 0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q_E \rightarrow 0.$$

If  $E$  and  $E^{\vee\vee}$  are both instantons, one immediately gets from the cohomology sequence that  $h^0(Q_E(-2)) = h^1(Q_E(-2)) = 0$ , so  $Q_E$  is a rank 0 instanton.

Dualizing (28) one obtains (27) using the fact that  $\mathcal{E}xt^1(Q_E, \mathcal{O}_{\mathbb{P}^3}) = 0$  (because  $Q_E$  is supported in codimension 2) and  $\mathcal{E}xt^2(E^{\vee\vee}, \mathcal{O}_{\mathbb{P}^3}) = 0$  (because  $E^{\vee\vee}$  is instanton).  $\square$

**3.3. Two examples.** We complete this section with two examples that highlight the necessity of the hypothesis used in Lemmata 10 and 11.

First, note that if  $E$  is a reflexive instanton sheaf that is not locally free, then  $\dim S_E = 0$  and  $S_E$  is not a rank 0 instanton. Moreover,  $E^\vee$  is not instanton.

Indeed, the first claim is clear, since the singular locus of reflexive sheaves must have codimension at least 3. Since  $\dim S_E = 0$ , then  $h^0(S_E(-2)) \neq 0$ , thus  $E^\vee$  is not instanton by Lemma 7, item (ii).

Here is a concrete example of a reflexive instanton sheaf which is not locally free, taken from [6, Example 5]. Consider the following instanton monad,

$$(29) \quad \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)$$

where  $\alpha$  and  $\beta$  are defined by

$$\alpha = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \\ 0 \\ x_3 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & 0 \end{pmatrix}.$$

Its cohomology sheaf is a rank 3 instanton sheaf, here denoted  $E$ , of charge 1. Its degeneracy locus, hence  $\text{Sing}(E)$ , consists of a single point, namely  $[0 : 0 : 0 : 1]$ , hence  $E$  is reflexive, but not locally free. In particular,  $E^\vee$  is not instanton.

We shall further discuss rank 3 instantons of charge 1 in Section 5.1 below.

Next, we provide an example of an instanton sheaf which is not reflexive and such that  $E^{\vee\vee}$  is not an instanton, taken from [6, Example 3].

Consider first the following linear monad,

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0;$$

according to [2, p. 5503], its cohomology sheaf is of the form  $\mathcal{I}_C(1)$ , the twisted ideal of a space curve  $C \hookrightarrow \mathbb{P}^3$ .

Floystad's result [2, Main Theorem] also guarantees, for any  $c \geq 1$ , the existence of a monad of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2c+4} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c+1} \rightarrow 0,$$

whose cohomology is a rank 3 locally free sheaf  $F$  with  $c_1(F) = -1$ .

The direct sum  $E := F \oplus \mathcal{I}_C(1)$  provides the desired example:  $E$  is a non reflexive instanton sheaf of rank 4 and charge  $c + 2$  ( $c \geq 1$ ), being the cohomology of the linear monad

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus c+2} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2c+8} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus c+2} \rightarrow 0.$$



However,  $E^{\vee\vee} = F^{\vee\vee} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$  is not an instanton sheaf, since  $H^0(E^{\vee\vee}(-1)) \simeq H^0(\mathcal{O}_{\mathbb{P}^3}) \neq 0$ . Note that  $Q_E \simeq \mathcal{O}_C(1)$ , thus  $Q_E$  may not be a rank 0 instanton. Furthermore, note also that  $E^\vee$  is locally free, but not an instanton either.

#### 4. SINGULAR LOCUS OF RANK 2 INSTANTON SHEAVES

We are finally in position to establish the main result of this paper.

Let  $E$  be a rank 2 instanton sheaf on  $\mathbb{P}^3$ . Applying Proposition 1 to its dual sheaf  $F = E^\vee$ , we conclude that  $E^{\vee\vee} \simeq E^\vee$ , since  $\det(E^\vee) = \mathcal{O}_{\mathbb{P}^3}$ . It then follows from Lemma 7 that both  $E^\vee$  and  $E^{\vee\vee}$  are instanton sheaves.

The fact that  $S_E$  and  $Q_E$  are rank 0 instantons follow from Lemma 10 and Lemma 11, respectively.

Furthermore,  $E^\vee$  and  $E^{\vee\vee}$  must be locally free by a direct application of Proposition 3, since  $h^2(E^\vee(-2)) = h^2(E^{\vee\vee}(-2)) = 0$ . In particular,  $S_{E^{\vee\vee}} = 0$ , thus  $S_E \simeq \mathcal{E}xt^2(Q_E, \mathcal{O}_{\mathbb{P}^3})$  by sequence (27). Since also  $Q_E \simeq \mathcal{E}xt^2(S_E, \mathcal{O}_{\mathbb{P}^3})$  by Lemma 8 item (ii), we have that  $Q_E$  and  $S_E$  are dual rank 0 instantons.

Finally, since  $\text{Sing}(E) = \text{Supp}(S_E) = \text{Supp}(Q_E)$ , it follows that  $\text{Sing}(E)$  has pure dimension 1.

Note, in addition, that if  $E$  has charge  $c$  and  $E^{\vee\vee}$  has charge  $c'$ , then  $Q_E$  and  $S_E$  are rank 0 instantons of degree  $d := c - c'$ . In fact, their Hilbert polynomials are given by  $P_{S_E}(k) = P_{Q_E}(k) = dk + 2d$ , so that  $Q_E$  may be regarded as points in the quot scheme  $\text{Quot}^{dk+2d}(E^{\vee\vee})$ .

To complete the proof of the Main Theorem, we assume that  $d = 1$ . Since  $Q_E$  is a rank 0 instanton of degree 1,  $Q_E(-1)$  admits a resolution of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow Q_E(-1) \rightarrow 0.$$

Therefore, in fact,  $Q_E(-1) \simeq \iota_* \mathcal{O}_\ell$  for some line  $\iota : \ell \hookrightarrow \mathbb{P}^3$ . In other words, the singular locus of  $E$  consists of a single line.

In particular, we also obtain the following claim.

**Corollary 12.** *Every rank 2 non locally free instanton sheaf  $E$  of charge 1 is of the form*

$$0 \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \rightarrow \iota_* \mathcal{O}_\ell(1) \rightarrow 0,$$

for some line  $\ell \in \mathbb{P}^3$ .

A complete classification of possible singular loci for rank 2 instanton sheaves of charge  $c$  seems to be a hard problem, since it requires an understanding of the quot schemes of rank 2 locally free instanton sheaves. A procedure to construct instanton sheaves with a prescribed singular locus is given in [7, Section 3]. To be precise, let  $E$  be an instanton sheaf of charge  $c$ , and consider triples  $(\Sigma, L, \varphi)$  for  $E$  consisting of the following:

- (i) an embedding  $\iota : \Sigma \hookrightarrow \mathbb{P}^3$  of a reduced, locally complete intersection curve of arithmetic genus  $g$  and degree  $d$ ;
- (ii) an invertible sheaf  $L \in \text{Pic}^{g-1}(\Sigma)$  such that  $h^0(\iota_* L) = h^1(\iota_* L) = 0$ ;
- (iii) a surjective morphism  $\varphi : E \rightarrow \iota_* L(2)$ .

It follows that  $F := \ker \varphi$  is an instanton sheaf of the same rank as  $E$  and charge  $c+d$ ; if  $E$  is locally free, then  $Q_F = \iota_* L(2)$ , so that the singular locus of  $F$  is precisely  $\Sigma$ . The difficulty, of course, is proving the existence of the surjective morphism  $\varphi$ . In [7], the cases of rational curves and elliptic quartic curves is considered.

## 5. SINGULAR LOCUS OF RANK 3 INSTANTON SHEAVES

In this section we show that instanton sheaf of rank larger than 2 may have 0-dimensional singularities, as well as singular loci which are not of pure dimension. These phenomena first occur for instanton sheaves of rank 3 and charge 1 and 2, respectively.

**5.1. Rank 3 instantons of charge 1.** We will now consider linear monads of the form

$$(30) \quad \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1).$$

Note that any surjective map  $\mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)$  may, after a linear change of homogeneous coordinates and a change of basis on the free sheaf  $\mathcal{O}_{\mathbb{P}^3}^{\oplus 5}$ , be written in the following form:

$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & 0 \end{pmatrix}.$$

It follows that the map  $\alpha$  is given by

$$(31) \quad \alpha = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \phi \end{pmatrix},$$

where  $\sigma_j \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$  must satisfy the monad equation

$$(32) \quad \sum_j x_j \sigma_j = 0.$$

Moreover, the injectivity of  $\alpha$  is equivalent to at least one of the sections  $\sigma_j, \phi$  being non-trivial.

However,  $\phi \equiv 0$  if and only if the monad (30) decomposes as a sum of two monads

$$(\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)) \bigoplus (0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0)$$

which in turns is equivalent to the cohomology of the (30) splitting as a direct sum  $E \oplus \mathcal{O}_{\mathbb{P}^3}$ , for some instanton sheaf  $E$  of rank 2 and charge 1.

We assume therefore, from now on, that  $\phi \neq 0$ . Let  $\Gamma$  be the subspace of  $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$  spanned by the sections  $\sigma_j$  and  $\phi$ ; in particular,  $\dim \Gamma \geq 1$ .

We observe that  $\dim \Gamma = 1$  if and only if the cohomology of the monad (30) splits as a sum  $\Omega_{\mathbb{P}^3}^1(1) \oplus \mathcal{O}_{\wp}$ , where  $\wp$  is the hyperplane defined by the equation  $\{\phi = 0\}$ . Indeed,  $\dim \Gamma = 1$  if and only if  $\sigma_j = \lambda_j \phi$  for each  $j = 1, 2, 3, 4$ ; thus one can find a basis for the free sheaf  $\mathcal{O}_{\mathbb{P}^3}^{\oplus 5}$  in which the map  $\alpha$  is of the form

$$\alpha = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \phi \end{pmatrix}.$$

It follows that the monad (30) decomposes as a sum of two linear monads

$$(0 \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)) \bigoplus (\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^3} \rightarrow 0)$$

which in turns is equivalent to the cohomology of the (30) splitting as a direct sum  $\Omega_{\mathbb{P}^3}^1(1) \oplus \mathcal{O}_{\wp}$ .

**Proposition 13.** *There exists a 1-1 correspondence between indecomposable instanton sheaves of rank 3 and charge 1 on  $\mathbb{P}^3$ , and non-trivial extensions of  $\mathcal{O}_\varphi$  by  $\Omega_{\mathbb{P}^3}^1(1)$ , for some hyperplane  $\varphi \subset \mathbb{P}^3$ .*

*Proof.* As seen above, every indecomposable rank 3 instanton sheaf  $E$  of charge 1 on  $\mathbb{P}^3$  is the cohomology of a monad of the form (30) with  $\phi \neq 0$  and  $\dim \Gamma \geq 2$ . The key observation here is that, in this case, the monad (30) can be written as a (non-trivial) extension of two simpler linear monads

$$(33) \quad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} & \xrightarrow{\omega} & \mathcal{O}_{\mathbb{P}^3}(1) \\ & & \downarrow & & \downarrow \simeq \\ \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\alpha} & \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(1) \\ \downarrow \simeq & & \downarrow & & \\ \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\cdot\phi} & \mathcal{O}_{\mathbb{P}^3} & & \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

where the map  $\omega$  is given by

$$\omega = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix},$$

and  $\alpha$  is given by (31).

Passing to cohomology, this short exact sequence of complexes yields precisely the short exact sequence

$$(34) \quad 0 \rightarrow \Omega_{\mathbb{P}^3}^1(1) \rightarrow E \rightarrow \mathcal{O}_\varphi \rightarrow 0$$

between their middle cohomologies, where  $\varphi$  is the hyperplane defined by the equation  $\phi = 0$ .

Conversely, if  $E$  is a non-trivial extension of  $\mathcal{O}_\varphi$  by  $\Omega_{\mathbb{P}^3}^1(1)$ , for some hyperplane  $\varphi \subset \mathbb{P}^3$ , then one can lift the short exact sequence (34) to a short exact sequence of complexes as in (33). From the considerations above, we know that the linear monad thus obtained in the middle row is such that  $\phi \neq 0$  and  $\dim \Gamma \geq 2$ , thus its cohomology sheaf is an indecomposable rank 3 instanton sheaf  $E$  of charge 1.  $\square$

The previous Proposition allows us to provide a neat description of the moduli space of indecomposable rank 3 instanton sheaves of charge 1, which we will denote here by  $\mathcal{I}^{\text{tf}}(3, 1)$ .

If  $E$  is such an object, let  $\varphi_E$  be the corresponding hyperplane in  $\mathbb{P}^3$ , obtained via the sequence (34); this yields a map

$$\varpi : \mathcal{I}^{\text{tf}}(3, 1) \rightarrow (\mathbb{P}^3)^\vee.$$

Next, note that  $\varpi$  is surjective. Indeed, consider first the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\cdot\phi} \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_\varphi \rightarrow 0,$$

i.e.  $\wp$  is the hyperplane within  $\mathbb{P}^3$  given by the equation  $\phi = 0$ . Applying the functor  $\text{Hom}(\cdot, \Omega_{\mathbb{P}^3}^1(1))$ , we conclude that

$$\text{Ext}^1(\mathcal{O}_{\wp}, \Omega_{\mathbb{P}^3}^1(1)) \simeq H^0(\Omega_{\mathbb{P}^3}^1(2)).$$

In particular, we have  $\dim \text{Ext}^1(\mathcal{O}_{\wp}, \Omega_{\mathbb{P}^3}^1(1)) = 6$ , independently of the hyperplane  $\wp$ . Thus for every hyperplane  $\wp \in (\mathbb{P}^3)^\vee$  one has non-trivial extensions by  $\Omega_{\mathbb{P}^3}^1(1)$ , and every such extension defines an element  $[E] \in \mathcal{I}^{\text{tf}}(3, 1)$  such that  $\varpi([E]) = \wp$ .

Furthermore, we also concluded that the fibres of  $\varpi$  are precisely the projectivization of  $\text{Ext}^1(\mathcal{O}_{\wp}, \Omega_{\mathbb{P}^3}^1(1))$ .

Summarizing the above discussion, we have the following result.

**Proposition 14.** *The moduli space  $\mathcal{I}^{\text{tf}}(3, 1)$  of indecomposable rank 3 instanton sheaves of charge 1 on  $\mathbb{P}^3$  is a projective variety of dimension 8, given by the total space of a  $\mathbb{P}^5$ -bundle over  $(\mathbb{P}^3)^\vee$ .*

We are finally able to characterize the singular loci of rank 3 instanton sheaves of charge 1.

**Proposition 15.** *The singular locus of a rank 3 instanton sheaf  $E$  of charge 1 which is not locally free is either a point, if  $E$  is reflexive, or a line, if  $E$  is not reflexive.*

*Proof.* First, let  $E$  be a reflexive instanton sheaf of rank 3 and charge 1. Then  $\dim \text{Sing}(E) = 0$ , and we must show that  $\text{Sing}(E)$  is a point.

Indeed, recall that  $\text{Sing}(E)$  coincides with the degeneration locus of the monad (30), which is given by the common zeros of the sections  $\sigma_j$  and  $\phi$ ; such set has dimension zero if and only if it consists of a single point.

Note, in addition, that  $E$  is reflexive if and only if  $\dim \Gamma = 3$ . We have already proved the only if part; conversely, if  $\dim \Gamma = 3$ , then  $\text{Sing}(E)$  consists of a single point, hence  $E$  must be reflexive by Lemma 6.

Now if  $E$  is indecomposable and not reflexive, then  $\dim \Gamma = 2$ , which means that the degeneration locus of the monad (30), and hence  $\text{Sing}(E)$ , is a line.

If, on the other hand,  $E$  is decomposable (i.e. if  $\dim \Gamma = 1$ ), then it decomposes as a sum  $E' \oplus \mathcal{O}_{\mathbb{P}^3}$  with  $E'$  being a non locally free rank 2 instanton of charge 1. It then follows from Proposition 12 that  $\text{Sing}(E')$ , which of course coincides with  $\text{Sing}(E)$ , is a line.

□

**Remark 16.** It is easy to see that every reflexive instanton sheaf of rank 3 on  $\mathbb{P}^3$  is  $\mu$ -semistable. Indeed, every instanton sheaf  $E$  on  $\mathbb{P}^3$  satisfies  $H^0(E(-1)) = H^0(E^\vee(-1)) = 0$ , thus  $\mu$ -semistability follows from the criterion in [12, Remark 1.2.6 b, page 167].

On the other hand, by [6, Prop. 16], there are no  $\mu$ -stable instanton sheaves of rank 3 and charge 1. Furthermore, one can check from (34) that  $h^0(E) = 1$ , and it follows that  $E$  is not (Gieseker) semistable either.

**5.2. Rank 3 instanton sheaves of charge 2.** In the last part of this paper, we present two interesting examples of rank 3 instanton sheaves of charge 2.

We begin by showing how to construct a rank 3 instanton sheaf of charge 2 whose singular locus is the disjoint union of a line and a point.

The starting point is an indecomposable reflexive instanton sheaf  $F$  of rank 3 and charge 1 which is not locally free. For instance, take the one obtained as cohomology of the monad (29); its singular locus is just the point  $P = [0 : 0 : 0 : 1]$ . One easily checks that it has trivial splitting type, that is, its restriction to a generic line is trivial.

Let  $\iota : \ell \hookrightarrow \mathbb{P}^3$  be a line in  $\mathbb{P}^3$  that does not contain the point  $[0 : 0 : 0 : 1]$  and to which the restriction of  $F$  is trivial. As we have checked above, the sheaf  $\iota_*\mathcal{O}_\ell(1)$  is a rank 0 instanton of degree 1. Moreover, since  $F|_\ell \simeq \mathcal{O}_\ell^{\oplus 2}$ , there are surjective maps  $\varphi : F \rightarrow \iota_*\mathcal{O}_\ell(1)$ ; let  $E := \ker \varphi$ .

From the short exact sequence

$$(35) \quad 0 \rightarrow E \rightarrow F \xrightarrow{\varphi} \iota_*\mathcal{O}_\ell(1) \rightarrow 0$$

one easily checks that  $E$  is a rank 3 instanton sheaf of charge 2. Indeed, since  $F$  is an instanton sheaf, it is easy to see that  $E$  is torsion free and that  $c_1(E) = c_3(E) = 0$  and  $c_2(E) = 2$ . One also checks immediately from the cohomology sequence that

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0.$$

**Remark 17.** The construction of the two previous paragraphs is a particular case of an *elementary transformation* of an instanton sheaf, as outlined in [7, Section 3].

Note also that  $E^\vee \simeq F^\vee$ , thus in particular  $E^{\vee\vee} \simeq F$ , so  $E^{\vee\vee}$  is also an instanton sheaf, and the sequence  $0 \rightarrow E \rightarrow E^{\vee\vee} \rightarrow Q_E \rightarrow 0$  matches the sequence (35), thus  $Q_E = \iota_*\mathcal{O}_\ell(1)$ . In addition, sequence (27) reduces to, in this case,

$$0 \rightarrow j_*\mathcal{O}_P \rightarrow S_E \rightarrow \iota_*\mathcal{O}_\ell(1),$$

where  $j$  denotes the inclusion of the point  $P = [0 : 0 : 0 : 1]$  within  $\mathbb{P}^3$ . It follows immediately that  $\text{Sing}(E) = P \cup \ell$ , which is not of pure dimension (neither 0 or 1).

We conclude with an explicit example of a monad whose cohomology is a reflexive instanton sheaf of rank 3 and charge 2 whose singular locus consists of two distinct points. We take

$$(36) \quad \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \xrightarrow{\alpha_{mn}} \mathcal{O}_{\mathbb{P}^3}^{\oplus 7} \xrightarrow{\beta_{mn}} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$$

with

$$\alpha_{mn} = \begin{pmatrix} x_2 & -x_4 \\ -m^2x_4 & x_2 \\ -x_1 & x_3 \\ n^2x_3 & -x_1 \\ -x_3 & \frac{1}{n}x_3 \\ -\frac{1}{m}x_3 - mx_4 & -\frac{1}{mn}x_3 - x_4 \\ x_3 - \frac{m}{n}x_4 & \frac{1}{m}(x_3 + x_4) \end{pmatrix}$$

and

$$\beta_{mn} = \begin{pmatrix} -x_1 & x_3 & -x_2 & x_4 & x_3 + x_4 & 0 & x_3 \\ n^2x_3 & -x_1 & m^2x_4 & -x_2 & x_4 & x_3 & \frac{1}{m}x_3 \end{pmatrix}$$

where  $n$  is a root of the equation  $n^3 + 2n^2 + n + 1 = 0$  and  $m = n + 1/n$  (this guarantees that  $\beta\alpha = 0$ ). One checks that  $\beta$  is surjective everywhere (since  $m \neq \pm 1$ ) and that  $\alpha$  fails to be injective only at the points  $[n : 0 : 1 : 0]$  and  $[0 : m : 0 : 1]$ .

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